JOURNAL OF APPROXIMATION THEORY 22, 27–32 (1978)

Approximation by Interpolating Polynomials

A. S. B. HOLLAND,* B. KUTTNER,†,‡ AND B. N. SAHNEY*

*Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada T2N 1N4 and
†Department of Pure Mathematics, University of Birmingham, Birmingham, England B15 2TT

Communicated by Richard S. Varga

Received April 30, 1976

The problem of convergence of interpolating polynomials of the type

$$I_n(x,f) = \frac{a_0^{(n)}}{2} + \sum_{k=1}^{n} (a_k^{(n)} \cos kt + b_k^{(n)} \sin kt)$$

with the interpolating points $t_j^{(n)} = 2\pi j/(2n+1)$ has been studied by A. Zygmund, who considered the partial sums of the interpolating polynomials $I_{n,\nu}(x,f)$. On the other hand, R. Taberski studied essentially the same problem, by considering the (C, α) summability of $I_{n,n}(x,f)$. Here a generalization of the results of Taberski is made by using a triangular matrix summability method.

1. INTRODUCTION

Let the real function $f(t)$ be integrable in the Riemann sense on $[0, 2\pi]$ and periodic with period 2π on $(-\infty, \infty)$.

Consider the triginometric polynomials

$$I_n(x,f) = \frac{a_0^{(n)}}{2} + \sum_{k=1}^{n} (a_k^{(n)} \cos kt + b_k^{(n)} \sin kt) \tag{1}$$

with interpolating points

$$t_j^{(n)} = 2\pi j/(2n+1); \qquad j = 0, \pm 1, \ldots, \pm n. \tag{2}$$

The problem of convergence of the interpolating polynomials (1) has been discussed by Zygmund [5], while (C, α) summability of the derivatives of (1) has been discussed by Taberski [4], for the cases $\alpha > 1$, $\alpha = 1$ and $0 < \alpha < 1$ separately. The object here is to unify the three cases and extend the results to the case of a matrix method $D = (d_{n,k})$.

‡ This work was supported by the National Research Council of Canada and Scientific Affairs Division of NATO.

0021-9045/78/0221-0027$02.00/0

Following Zygmund [5], we write the partial sum of the interpolating trigonometric polynomial as

$$I_{n,\nu}(x, f) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)\, D_\nu(t - x)\, d\phi_n(t)$$

$$= \frac{1}{\pi} \int_{x-\pi}^{x+\pi} f(t)\, D_\nu(t - x)\, d\phi_n(t),$$

where

$$D_\nu(t) = \frac{1}{2} + \sum_{k=1}^{\nu} \cos kt = \frac{\sin(\nu + \frac{1}{2})t}{2 \sin \frac{1}{2}t}$$

and $\phi_n(t)$ is a step function equal to $2\pi j/(2n + 1)$ for

$$t \in [t_{j-1}^{(n)}, t_j^{(n)}); \qquad j = 0, \pm 1, \ldots .$$

We write

$$\int_a^b g(t)\, d\phi_n(t) = \frac{2\pi}{2n + 1} \sum_{j=\alpha}^{\beta} g(t_j^{(n)}) \tag{3}$$

for a function defined on $(a, b]$ and

$$a < t_\alpha^{(n)} < t_{\alpha+1}^{(n)} < \cdots < t_\beta^{(n)} = b.$$

Now if g is of period 2π, then

$$\int_\alpha^{\alpha+2\pi} g(t)\, d\phi_n(t) \tag{4}$$

is independent of α.

2. Matrix Transform and Notations

Let $D = (d_{n,k})$ be a regular triangular sequence-to-sequence transformation. Write $K_n(t)$ for the kernel,

$$K_n(t) = \sum_{k=0}^{n} d_{n,k} \frac{\sin(k + \frac{1}{2})t}{\sin \frac{1}{2}t}\,. \tag{5}$$

Suppose that D is such that for large ν, uniformly in $n \geqslant \nu$, $0 < c \leqslant 1$, $2\pi/(2\nu + 1) < \eta < \delta \leqslant \pi$, we have

$$\sum K_\nu' \left\{(c + \rho) \frac{2\pi}{2n + 1}\right\} = O\{n\nu R(\nu\eta)\}, \tag{6}$$

where the sum in the last expression is taken over $\eta < (c+\rho)(2\pi/(2n+1)) \leqslant \delta$. Furthermore, suppose $R(n)$ is positive, decreasing, and such that

$$\int_1^\infty R(n)\, dn \tag{7}$$

converges. $R(n)$ may be considered as a remainder term.

3. Proof of Theorem.

THEOREM. *Suppose $f(t)$ is Riemann integrable on $(-\pi, \pi)$, absolutely continuous for all $x \in [a, b] \subset (-\pi, \pi)$ and such that*

$$\int_x^{x+h} |f'(t) - f'(x)|\, dt = o(h), \tag{8}$$

then $\sum_{k=0}^{\nu} d_{\nu,k} I'_{n,k}(x, f) \to f'(x)$ as $\nu \to \infty$, uniformly in n for $n \geqslant \nu$, i.e., the sequence of differentiated interpolation poylnomials is D-approximable at x, to $f'(x)$, where D is defined in Section 2.

Proof. Let

$$F_x(t) = f(t) - f(x) - f'(x)\sin(t - x)$$

and

$$\Phi(t) = \int_x^t |F_x'(t)|\, dt = o(|\, t - x\,|). \tag{9}$$

Then it is enough to show that

$$\int_{-\pi}^{\pi} F_x(t)\, K_\nu'(t - x)\, d\phi_n(t) \to 0, \qquad \text{as} \quad \nu \to \infty \tag{10}$$

uniformly in n. Here, K_ν' denotes differentiation with respect to the argument.

In view of (4) and the absolute continuity of $f(t)$, in $(x - \delta, x + \delta)$ for δ small enough, the contribution of the integrals

$$\int_{-\pi}^{x-\delta}, \int_{x+\delta}^{\pi} \qquad \text{to} \qquad I'_{n,\nu}(x, f)$$

tend to zero as $\nu \to \infty$ uniformly in $n \geqslant \nu$, (see Zygmund [5]). Thus we can consider the case of the integral over $[x - \delta, x + \delta]$ only.

We take the integral over $[x, x + \delta]$ only, since the integral over $[x - \delta, x]$ follows similarly.

We know that

$$K_\nu'(t) = \frac{d}{dt}(K_\nu(t)) = \frac{d}{dt}\left\{\sum_{k=0}^{\nu} d_{\nu,k} \frac{\sin(k+\frac{1}{2})t}{\sin\frac{1}{2}t}\right\}$$

$$= O\left\{\sum_{k=0}^{\nu} (k+1)^2 \mid d_{\nu,k} \mid\right\}$$

$$= O\left\{\nu^2 \sum_{k=0}^{\nu} \mid d_{\nu,k} \mid\right\}$$

$$= O(\nu^2)$$

since D is regular.

The jumps of $\phi_n(t)$ in the interval $(x, x+\delta)$ occur at intervals of $2\pi/(2n+1)$; let them occur at

$$t = x + (c+\rho)(2\pi/(2n+1)), \qquad (\rho = 0, 1, 2,...)$$

and $0 < c \leqslant 1$.

Let us first consider

$$\int_x^{x+(2\pi/(2\nu+1))} F_x(t)\, K_\nu'(t-x)\, d\phi_n(t).$$

$\phi_n(t)$ is a step function in this interval and is bounded by $O(1/n)$. Thus we have the following: The number of jumps is $O(n/\nu)$, the magnitude of each jump of $\phi_n(t)$ is $O(1/n)$ and we have, uniformly, $K_\nu'(t-x) = O(\nu^2)$. Also, by (9), we have, uniformly in the range considered,

$$F_x(t) = o(1/\nu).$$

Hence

$$\int_x^{x+(2\pi/(2\nu+1))} F_x(t)\, K_\nu'(t-x)\, d\phi_n(t) = o(1).$$

Now consider

$$\int_{x+(2\pi/(2\nu+1))}^{x+\delta}.$$

Let us write

$$A_{n,\nu}(x) \equiv \int_{x+(2\pi/(2\nu+1))}^{x+\delta} F_x(t)\, K_\nu'(t-x)\, d\phi_n(t).$$

Then

$$A_{n,\nu}(x) = \frac{2\pi}{2n+1} \sum F_x\left(x + (c+\rho)\frac{2\pi}{2n+1}\right) K_\nu'\left\{(c+\rho)\frac{2\pi}{2n+1}\right\}$$

$$= \frac{2\pi}{2n+1} \sum K_\nu'\left\{(c+\rho)\frac{2\pi}{2n+1}\right\} \int_x^{x+(c+\rho)(2\pi/(2n+1))} F_x'(t)\, dt,$$

where the last two summations are taken over $2\pi/(2\nu+1) < (c+\rho) \times 2\pi/(2n+1) \leqslant \delta$, and thus

$$A_{n,\nu}(x) \leqslant \frac{2\pi}{2n+1} \int_x^{x+\delta} F_x'(t) \sum K_\nu' \left\{(c+\rho)\frac{2\pi}{2n+1}\right\},$$

where the sum is over $\max(t-x, 2\pi/(2\nu+1)) < (c+\rho)(2\pi/(2n+1)) \leqslant \delta$. Thus

$$A_{n,\nu}(x) = O\left\{\frac{1}{n} R\left(\frac{2\pi\nu}{2\nu+1}\right) n\nu \int_x^{x+(2\pi/(2\nu+1))} |F_x'(t)|\, dt\right\}$$

$$+ O\left\{\frac{1}{n} n\nu \int_{x+(2\pi/(2\nu+1))}^{x+\delta} |F_x'(t)|\, R(\nu(t-x))\, dt\right\} \qquad (11)$$

by (6).

Since $R(\nu)$ is decreasing for all ν, $R(2\pi\nu/(2\nu+1)) \leqslant R(\pi) =$ constant; thus it follows from (9) that the first term on the right side of (11) is

$$O\{\nu O(1)\, o(2\pi/(2\nu+1))\} = o(1).$$

We now consider the second term in (11) and on integrating by parts,

$$\begin{aligned} &\nu \int_{x+(2\pi/(2\nu+1))}^{x+\delta} |F_x'(t)|\, R(\nu(t-x))\, dt \\ &\quad = \nu \int_{x+(2\pi/(2\nu+1))}^{x+\delta} R(\nu(t-x)\, d\Phi(t) \\ &\quad = \nu R(\nu\delta)\, \Phi(\delta) - \nu R\left(\frac{2\pi\nu}{2\nu+1}\right) \Phi\left(x+\frac{2\pi}{2\nu+1}\right) \\ &\qquad - \nu \int_{x+(2\pi/(2\nu+1))}^{x+\delta} \Phi(t)\, d_t R(\nu(t-x)). \end{aligned} \qquad (12)$$

Since $R(x)$ is decreasing for all x, the assumption that $\int_1^\infty R(x)\, dx < \infty$ implies that $R(\nu\delta) = o(1/\nu)$, so that the first term in (12) is $o(1)$. Also, since $R(2\pi\nu/(2\nu+1)) = O(1)$, the second term in (12) is $o(1)$.

Now given any $\epsilon > 0$, there is an η, such

$$|\Phi(t)| < \epsilon(t-x)$$

for $t - x < \eta$.

Also, there is some constant A, such that

$$\Phi(t) \leqslant A(t-x)$$

in $(x, x+\delta)$. Putting $\nu(t-x) = u$, we observe that the third term in (12) does not exceed

$$\epsilon \int_{2\pi\nu/(2\nu+1)}^{\eta\nu} u\, |dR(u)| + A \int_{\eta\nu}^{\delta\nu} u\, |dR(u)|.$$

The result will follow if we prove that

$$\int_1^\infty u \mid dR(u)\mid < \infty.$$

However, since $R(n) = o(1/n)$, then using integration by parts, we have

$$\int_1^\infty u \mid dR(u)\mid < \infty$$

and thus (10) is true and hence the theorem is proved.

Note. It may be remarked that in the case when D is (C, α), the hypothesis of Taberski's Theorem 2 [4], is satisfied with the remainder $R(u)$ given by $u^{-1-\alpha}$. Thus his result is a particular case of the above theorem.

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